IInd Midsemestral exam 2004
Algebra IV: B.Sury

1. Let M, N be left A-modules and suppose N is semisimple. If $\alpha, \beta : M \to N$ are in $\mathrm{Hom}_A(M, N)$ such that Ker $\alpha \subseteq$ Ker β, show that there exists $\theta \in 0$ $\mathrm{End}_A(N)$ satisfying $\beta = \theta \ o \ \alpha$.

2. Let A be any commutative ring and let G be a finite group. Show that the group ring $A[G]$ is left Noetherian (that is, any ascending chain of left ideals is finite) if, and only if, it is right Noetherian. You may use the map $\sum_g a_g g \mapsto \sum_g a_g g^{-1}$.

3. Let G be a finite group and $f, g : G \to \subseteq$ be class functions. Prove Plancherel's formula : $< f, g >= \sum\limits_{i=1}^{s} < f, \mathcal{X}_i >< \mathcal{X}_i, g >$ where $\mathcal{X}_1, \ldots, \mathcal{X}_s$ are the irreducible characters of G.

4. Consider the following character table of a finite group (where $\omega = e^{2\pi i/3}$):

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
$\mathcal{X}_{p1}$	1	1	1	1	1	1	1
$\mathcal{X}_{p2}$	1	1	1	ω^2	ω	ω^2	ω
$\mathcal{X}_{p3}$	1	1	1	ω	ω^2	ω	ω^2
$\mathcal{X}_{p4}$	2	-2	0	-1	-1	1	1
$\mathcal{X}_{p5}$	2	-2	0	-ω^2	-ω	ω^2	ω
$\mathcal{X}_{p6}$	2	-2	0	-ω	-ω^2	ω	ω^2
$\mathcal{X}_{p7}$	3	3	-1	0	0	0	0

Find the order of the group and cardinalities of the conjugacy classes.

5. If a finite group has exactly three irreducible complex representations, prove that it is isomorphic either to $\mathbb{Z}/3\mathbb{Z}$ or to S_3.

6. Let K be algebraically closed and suppose $G \subseteq GL_n(K)$ is a finite group which is completely reducible. Prove that there exists $P \in GL_n(K)$ such that PAP^{-1} is a diagonal matrix for all $A \in G$

7. Prove that every simple ring must be of the form $M_n(D)$ for some division ring D and some n.

8. Let A be a left Artinian ring (that is, every decreasing chain of left ideals is finite). If the Jacobson radical Jac(A) (the intersection of all maximal left ideals) is zero, prove that A is left semisimple.

9. $G \subseteq GL_n(\subset)$ be a finite group such that for some $r \geq 1, \sum_g (tr(g))^r = 0$. Prove that $\sum_g g_{11}^r = 0$ where